

Bounded – Yes, but 4?

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Abstract

In this paper we will examine unions of oriented and non-oriented unit squares in same plane and measure the ratio of perimeter to area of these unions. In 1998, T. Keleti published the conjecture that this ratio never exceeds 4. We outline the current state of research on this conjecture and give two proofs of a special case. Finally, we explore the difficulties that arise from using similar methods in the general case and examine properties of any potential counterexample.

1 Introduction

The purpose of this paper is to introduce Tamas Keleti’s *Perimeter to Area Conjecture*, *PAC*. This conjecture concerning unions of unit squares in the Euclidean plane, \mathbb{R}^2 , is at the same time, easy and elementary to state, but elusive to penetrate. As with other such *elementary* statements, the fact that it has defied solution shows that we don’t understand something quite fundamental about basic geometry in a very familiar ambient setting. Here we outline the current state of knowledge about the problem and in Section 7 use the isoperimetric inequality to provide some new insight concerning a potential counterexample. The conjecture itself is this:

Keleti’s Perimeter to Area Conjecture. *The perimeter to area ratio of the union of finitely many unit squares in a plane does not exceed 4.*

The problem of showing that this ratio is simply bounded at all let alone by 4 first seems to have appeared as *Problem 6* on the famous Hungarian *Schweitzer Competition* in 1998 [8]. Even that is not completely trivial, though several Hungarian undergraduates managed a proof for the competition. Later that same year, Keleti published his *Perimeter to Area Conjecture* that this bound is actually 4. To date, the best known bound is slightly less than 5.6. This bound was found by Keleti’s student Zoltán Gyenes in his master’s thesis [4]. The *PAC* is particularly intriguing as some of its obvious generalizations are false. In particular, Gyenes, also in [4], showed that a corresponding conjecture is false if “square” is replaced by “convex set” even if the union consists of two congruent convex sets. The example is disarmingly simple. Let E_1 denote a

unit square centered at the origin with a small isosceles triangle “clipped” from one corner, see Figure 1 below.



If x denotes the height of that clipped triangle, then in terms of x , the perimeter and area of E_1 are respectively $p(x) = 4 - x(2 - \sqrt{2})$ and $a(x) = 1 - x^2/2$. From this it is easy to compute that the derivative of the perimeter to area ratio is negative when $x = 0$ and so for small values of x , the perimeter to area ratio of E_1 is less than 4. But if E_2 denotes the set obtained by rotating E_1 by π about the origin, then $E_1 \cup E_2$ is simply the original unit square whose perimeter to area ratio is obviously exactly 4. This is the example referred to below.

Gyenes’s Example. *There exist congruent convex sets , $E_1 \cong E_2 \subset \mathbb{R}^2$ such that the perimeter to area ratio for $E_1 \cup E_2$ exceeds the perimeter to area ratio for either one of them.*

While the *PAC* as stated remains unresolved, a special case of it is known.

Theorem 1. *The perimeter to area ratio of the union of finitely many axis oriented unit squares in a plane does not exceed 4.*

Gyenes gives a proof of this theorem in [4]. In Section 3 we outline Gyenes’s proof of Theorem 1 and show how he obtains the bound of 5.6 for the general case. Two additional elementary proofs for Theorem 1 are given in Sections 4 and 5. In Section 6.2 we examine the special case of non- oriented squares centered at the same point. In Section 7 we use the isoperimetric inequality to find strict conditions which any optimal counterexample must satisfy.

2 Notation

Throughout this paper, $\mathcal{H} = \bigcup_{i=1}^n H_i$ will be the union of finitely many unit squares H_i in \mathbb{R}^2 . By a *vertex* of \mathcal{H} we mean a point p on the boundary of \mathcal{H} which lies on the boundary of more than one of the H_i . The perimeter function $p(\cdot)$ takes a closed, bounded polygonal figure in the plane as input and returns that figure’s perimeter. The area function $a(\cdot)$ takes a closed, bounded polygonal figure in the plane as input and returns that figure’s area. The value Δp refers to the change in perimeter under a given action (adding a square, removing

a square, moving a square, etc.). When adding a square H_{n+1} , $\Delta p = p(H \cup H_{n+1}) - p(H)$. When subtracting the square H_n , that is $\Delta p = p(H) - p(H \setminus H_n)$, the function Δa is defined analogously. Finally, if A is a region in \mathbb{R}^2 , then ∂A will denote the boundary of A . Other notation will be defined as needed.

3 Gyenes's *PAC* Results

That the perimeter–area ratio for squares is bounded by 5.6 and that the bound is exactly 4 in the special case of axis oriented squares both follow from a general theorem obtained by Gyenes in [4] on the surface-area to volume ratio of the union of finitely many copies of a fixed set in \mathbb{R}^n . Without giving details or even complete background definitions, we state this result below to highlight the nature of the general theorem. Here, $T_{A,\mu}$ is a measure of the thinness of the set A as measured via a fixed probability measure μ ; see [4, Section 2] for details.

Gyenes's Polyhedral Theorem. *If \mathcal{H} is the union of a finite set H_i of congruent polyhedra in \mathbb{R}^n , then for any fixed probability measure μ , the ratio of the surface-area of \mathcal{H} to the volume of \mathcal{H} does not exceed $\frac{1}{T_\mu}$, where T_μ is the infimum of the set $\{T_{A,\mu} : A \in H_i\}$.*

Stripping the broader theorem of its generality, we first present Gyenes's proof of Theorem 1 and we then show how his 5.6 bound is obtained for non-oriented squares.

Gyenes's Proof of Theorem 1. Let $\{H_i : i = 1, 2, \dots, n\}$ be a finite collection of unit squares in \mathbb{R}^2 , suppose that the edges of each H_i are either vertical or horizontal, and set $\mathcal{H} = \cup_{i=1}^n H_i$. To begin, fix i and a non-vertex point $p \in \partial H_i$. Let $\Theta \equiv \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. For each $\theta \in \Theta$, let $l_{p,H_i}(\theta)$ denote the length of the line segment in the interior of H_i that begins at p and is in direction θ . In the case of a single oriented unit square, this length is 1 in the direction that goes directly across the square and 0 in the other three cardinal directions. Hence, for a fixed square H_i and fixed non-vertex point $p \in \partial H$, the sum of $l_{p,H_i}(\theta)$ over the four cardinal directions is simply

$$\sum_{\theta \in \Theta} l_{p,H_i}(\theta) = 1. \quad (1)$$

Now, consider the entire figure \mathcal{H} and partition $\partial \mathcal{H}$ into finitely many line segments s_j such that

1. each s_j is contained entirely in the boundary a single $H_i \equiv H_{i(j)}$,
2. the s_j 's are disjoint except for possibly at their endpoints, and
3. $\bigcup_j s_j = \partial \mathcal{H}$.

Let $|s_j|$ denote the length of segment s_j and fix $\theta \in \Theta$. We bound the area of \mathcal{H} below as the sum of the areas of the rectangular strips having one side s_j (and the other side either 1 or 0), to obtain:

$$a(\mathcal{H}) \geq \sum_j |s_j| l_{M_j, H_i}(\theta),$$

where M_j is the midpoint of s_j and $H_i = H_{i(j)}$ is the square with $s_j \subset \partial H_i$.

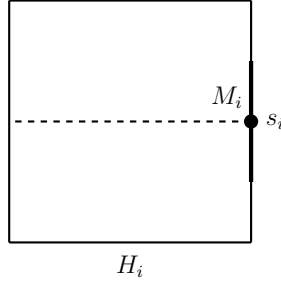


Figure 1: The Geometry in H_i with $\theta = \frac{3\pi}{2}$

Averaging across the four cardinal directions and using (1) then yields:

$$a(\mathcal{H}) \geq \frac{1}{4} \sum_{\theta \in \Theta} \sum_j |s_j| l_{M_j, H_i}(\theta) = \frac{1}{4} \sum_j |s_j| \sum_{\theta \in \Theta} l_{M_j, H_i}(\theta) = \frac{1}{4} \sum_i |s_i|.$$

Hence, $\frac{1}{4} \sum_i |s_i| \leq a(\mathcal{H})$. However, $\sum_i |s_i|$ is exactly $p(\mathcal{H})$ and therefore, $\frac{1}{4} p(\mathcal{H}) \leq a(\mathcal{H})$ and $\frac{p(\mathcal{H})}{a(\mathcal{H})} \leq 4$, as desired. \square

Gyenes obtains his bound of 5.6 for the case of non-oriented squares using an area finding integral and Fubini's Theorem. Again, we present the Gyenes proof, but restrict the scope to the squares in \mathbb{R}^2

The Gyenes Bound. Let $\{H_i : i = 1, 2, \dots, n\}$ and $\mathcal{H} = \cup_{i=1}^n H_i$ be as in the previous proof and let $\theta \in [0, 2\pi)$ be fixed. Define the *thickness* in the direction $\theta \in [0, 2\pi)$ at a point $p \in \partial \mathcal{H}$ to be

$$\tau(p, \theta) := l_{p, \mathcal{H}}(\theta) \sin(\phi)$$

where $l_{p, \mathcal{H}}(\theta)$ is the length of the line segment transversing the interior of \mathcal{H} that begins at p and is in direction θ . The angle $\phi = \phi(p, \theta)$ is the smaller of the two angles that this line segment makes with $\partial \mathcal{H}$ at p and is well defined except at vertices of \mathcal{H} . See Figure 2.

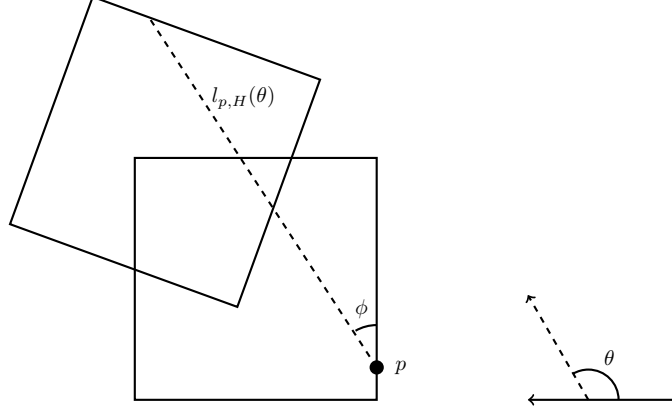


Figure 2: Thickness at a Boundary Point

As \mathcal{H} is the finite union of unit squares, it follows that for a fixed θ , the function $l_{p,H}(\theta)$ is well defined on $\partial\mathcal{H}$ and is piecewise linear. Hence, $\tau(p, \theta)$ is well defined and piecewise linear except at the vertices of \mathcal{H} . The projection of \mathcal{H} onto a line of direction $\theta + \frac{\pi}{2}$ is a finite union of non-degenerate closed intervals, and hence, the area of \mathcal{H} , $a(\mathcal{H})$ is the (Riemann) integral of $l_{p,H}(\theta)$ over that projection. Changing variables then to integrate around the boundary of \mathcal{H} , we conclude that for each fixed $\theta \in [0, 2\pi)$

$$a(\mathcal{H}) = \int_{p \in \partial H} \tau(p, \theta) ds.$$

where s is the arc-length parameterization of $\partial\mathcal{H}$. Averaging over $[0, 2\pi]$ and applying Fubini yields

$$\begin{aligned} a(H) &= \frac{1}{2\pi} \int_0^{2\pi} \int_{p \in \partial H} \tau(p, \theta) ds d\theta \\ &= \frac{1}{2\pi} \int_{p \in \partial H} \int_0^{2\pi} \tau(p, \theta) d\theta ds. \end{aligned} \tag{2}$$

Each non-vertex point, $p \in \partial\mathcal{H}$ is also on the boundary of a unique component unit square, $H_{i(p)}$. We let $l_{p,\mathcal{H}}^*(\theta) = l_{p,H_{i(p)}}(\theta) \leq l_{p,\mathcal{H}}(\theta)$ and $\tau(p, \theta) := l_{p,H}^*(\theta) \sin(\phi)$. Since $\sin(\phi) \geq 0$, it follows that $\tau(p, \theta) \geq \tau^*(p, \theta)$ for all non-vertex points $p \in \partial\mathcal{H}$. See Figure 3.

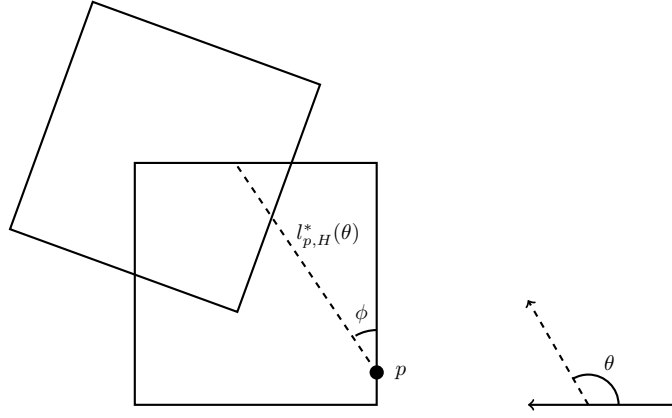


Figure 3: Less Thickness at a Boundary Point

Now let $T^* = \inf \int_0^{2\pi} \tau^*(p, \theta) d\theta$ where the inf is taken over the non-vertex points of $H_{i(p)}$. Then, T^* is independent of p (or $i(p)$) and so combining the inequality $\tau \geq \tau^*$ with the definition of T^* and equation (2) we obtain:

$$\begin{aligned}
 a(H) &= \frac{1}{2\pi} \int_{p \in \partial \mathcal{H}} \int_0^{2\pi} \tau(p, \theta) d\theta ds \\
 &\geq \frac{1}{2\pi} \int_{p \in \partial \mathcal{H}} \int_0^{2\pi} \tau^*(p, \theta) d\theta ds \\
 &\geq \frac{1}{2\pi} \int_{p \in \partial \mathcal{H}} T^* = \frac{T^*}{2\pi} \cdot p(\mathcal{H}).
 \end{aligned} \tag{3}$$

Or,

$$\frac{p(\mathcal{H})}{a(\mathcal{H})} \leq \frac{2\pi}{T^*}. \tag{4}$$

Finding T^* is just a matter of computation. First,

$$l^*(p, \theta) = \begin{cases} (1-x) \sec(\theta) & 0 \leq \theta \leq \cot^{-1}(1-x) \\ \csc(\theta) & \cot^{-1}(1-x) \leq \theta \leq \pi - \cot^{-1}(x) \\ -x \sec(\theta) & \pi - \cot^{-1}(x) \leq \theta \leq \pi \\ 0 & \pi \leq \theta < 2\pi. \end{cases}$$

See Figure 4.

Figure 4: Computing $l^*(p, \theta)$

In this normalized setting $\theta = \phi$ so that

$$\tau^*(p, \theta) = \begin{cases} (1-x)\tan(\theta) & 0 \leq \theta \leq \cot^{-1}(1-x) \\ 1 & \cot^{-1}(1-x) \leq \theta \leq \pi - \cot^{-1}(x) \\ -x\tan(\theta) & \pi - \cot^{-1}(x) \leq \theta \leq \pi \\ 0 & \pi \leq \theta < 2\pi, \end{cases}$$

and hence,

$$\int_0^{2\pi} \tau^*(p, \theta) d\theta = \frac{\ln(1 + (1-x)^2)}{2} - \ln(1-x) + \pi - \cot^{-1}(1-x) - \cot^{-1}(x).$$

This function is increasing over $[0, 1]$ so that the minimum is $\frac{2\pi}{\frac{1}{2}\ln 2 + \frac{\pi}{4}}$ occurring at $x = 0$. Hence, $\frac{p(\mathcal{H})}{a(\mathcal{H})} \leq \frac{2\pi}{\frac{1}{2}\ln 2 + \frac{\pi}{4}} \leq 5.6$. \square

Details of the more general theorem can be found in [4].

4 The Bump Method

In this section we present the first of two additional proofs of Theorem 1. This proof relies on the following elementary fact about rectangles.

Lemma 2. *Let $R \subset [0, 1]^2$ be a rectangle. Then $\frac{p(R)}{a(R)} \geq 4$.*

Proof. If R is an “a” by “b” rectangle, then as $R \subset [0, 1]^2$, $0 \leq a, b \leq 1$. Hence, $a(b-1) + b(a-1) \leq 0$, and the desired inequality immediately follows. \square

Proof of Theorem 1 - The Bump Method. Suppose that $\mathcal{H} = \cup_{i=1}^n H_i$ where H_i is an oriented unit square (vertical and horizontal edges) for $i = 1, 2, \dots, n$. We induct on the number of squares in \mathcal{H} . Clearly, $\frac{p(\mathcal{H})}{a(\mathcal{H})} = 4$ if \mathcal{H} is a single square. Suppose that \mathcal{H} consists of n squares and that $\frac{p(\mathcal{H})}{a(\mathcal{H})} \leq 4$. We show that adding another square will not cause the ratio to exceed 4. That is, no matter how the square H_{n+1} is added to the figure,

$$\frac{p(\mathcal{H} \cup H_{n+1})}{a(\mathcal{H} \cup H_{n+1})} \leq 4$$

Note that

$$\frac{p(\mathcal{H} \cup H_{n+1})}{a(\mathcal{H} \cup H_{n+1})} = \frac{p(\mathcal{H}) + \Delta p}{a(\mathcal{H}) + \Delta a},$$

and because $\frac{p(\mathcal{H})}{a(\mathcal{H})} \leq 4$, it is sufficient to show that $\frac{\Delta p}{\Delta a} \leq 4$. To calculate $\frac{\Delta p}{\Delta a}$, one only needs to examine how H_{n+1} intersects the original figure. We identify at most four disjoint rectangles in H_{n+1} that enable us to use Lemma 2 to finish the proof.

If $\mathcal{H} \cap H_{n+1} = \emptyset$ (or is a single point), then $\frac{\Delta p}{\Delta a} = \frac{p(H_{n+1})}{a(H_{n+1})} = 4$ and we are finished.

Hence, we may assume that $\mathcal{H} \cap \text{interior}(H_{n+1}) \neq \emptyset$. Since \mathcal{H} is comprised of oriented unit squares, and $\mathcal{H} \cap H_{n+1} \neq \emptyset$ it follows that $\text{interior}(\mathcal{H})$ contains at least one vertex of H_{n+1} . There are several cases depending on the number of vertices of H_{n+1} that are in \mathcal{H} .

Case 1 $\mathcal{H} \cap H_{n+1}$ contains exactly one vertex of H_{n+1} .

We suppose that the bottom left vertex of H_{n+1} , denoted v is at the origin and that $v \in \mathcal{H}$. Suppose (x_r, y_r) and (x_s, y_s) are the respective rightmost and topmost points of $\mathcal{H} \cap H_{n+1}$.

We “bump out” $\mathcal{H} \cap H_{n+1}$ by replacing $\mathcal{H} \cap H_{n+1}$ with the rectangle RB whose vertices are $v = (0, 0)$, $(x_r, 0)$, $(0, y_s)$, and (x_r, y_s) . See Figure 5.

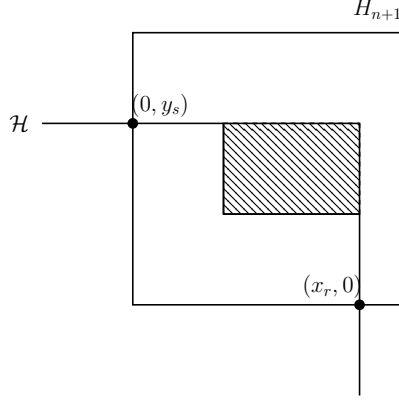


Figure 5: “Bumping Out” within H_{n+1}

For notational simplicity, denote the “stair-step” region $\mathcal{H} \cap H_{n+1}$ by SS . Denote the portion of the $\partial(SS)$ which also lies on the boundary of H_{n+1} by γ_1 and the remainder of $\partial(SS)$ by γ_2 . Then,

$$\begin{aligned} \frac{\Delta p}{\Delta a} &= \frac{p(\mathcal{H} \cup H_{n+1}) - p(\mathcal{H})}{a(\mathcal{H} \cup H_{n+1}) - a(\mathcal{H})} = \frac{(p(\mathcal{H}) - \ell(\gamma_2) + 4 - \ell(\gamma_1)) - p(\mathcal{H})}{(a(\mathcal{H}) - a(SS) + 1) - a(\mathcal{H})} \\ &= \frac{4 - \ell(\gamma_1) - \ell(\gamma_2)}{1 - a(SS)} = \frac{4 - p(RB)}{1 - a(SS)} \leq \frac{4 - p(RB)}{1 - a(RB)} \end{aligned}$$

By Lemma 2, $\frac{p(RB)}{a(RB)} \geq 4$ and hence, $\frac{4 - p(RB)}{1 - a(RB)} \leq 4$ completing the proof.

Case 2 $\mathcal{H} \cap H_{n+1}$ contains exactly two vertices, v_1 and v_2 of H_{n+1} .

We first “bump out” $\mathcal{H} \cap H_{n+1}$ as in Case 1, but this time we obtain two rectangular regions, RB_1 at v_1 and RB_2 at v_2 . If $RB_1 \cap RB_2 = \emptyset$, then using analogous estimates as in Case 1 we find that

$$\frac{\Delta p}{\Delta a} = \frac{p(\mathcal{H} \cup H_{n+1}) - p(\mathcal{H})}{a(\mathcal{H} \cup H_{n+1}) - a(\mathcal{H})} \leq \frac{4 - p(RB_1) - p(RB_2)}{1 - a(RB_1) - a(RB_2)}.$$

Now, from Lemma 2 we know that both $\frac{p(RB_1)}{a(RB_1)} \geq 4$ and $\frac{p(RB_2)}{a(RB_2)} \geq 4$ and hence, it follows directly that $\frac{4 - p(RB_1) - p(RB_2)}{1 - a(RB_1) - a(RB_2)} \leq 4$ as desired.

If $RB_1 \cap RB_2 \neq \emptyset$, there are two subcases to consider depending on whether v_1 and v_2 are adjacent or diagonally opposite one another.

In the case that v_1 and v_2 are adjacent and RB_1 overlaps RB_2 , we “bump out” the region $RB_1 \cup RB_2$ to the smallest oriented rectangle containing

$RB_1 \cup RB_2$ and denote that rectangle by RB^* . See Figure 6. The computation we did in Case 1 now applies to this situation and we compute that

$$\frac{\Delta p}{\Delta a} = \frac{p(\mathcal{H} \cup H_{n+1}) - p(\mathcal{H})}{a(\mathcal{H} \cup H_{n+1}) - a(\mathcal{H})} \leq \frac{4 - p(RB^*)}{1 - a(RB^*)}.$$

An application of Lemma 2 completes this case.

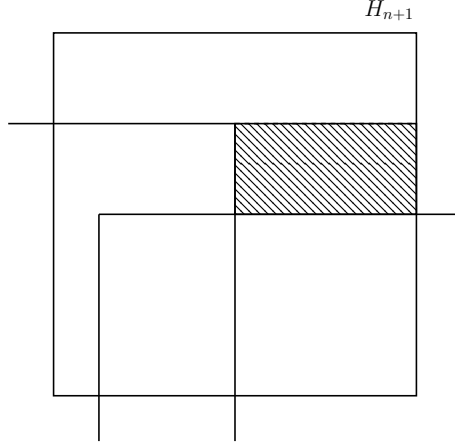


Figure 6: Another “Bumping Out” within H_{n+1}

Finally, if v_1 and v_2 are diagonally opposite and RB_1 overlaps RB_2 , then $\Delta p \leq 0$ so that the conclusion trivially holds.

This, then completes Case 2. The cases where $\mathcal{H} \cup H_{n+1}$ contains exactly three or exactly four points are completely analogous, and hence this also completes the “Bump Method” proof of Theorem 1. \square

5 The Method of Boundary Strips

Proof 3 of Theorem 1 - Boundary Strips. This proof is similar to Gyenes’s in Section 3 in that we add up the area of strips along the boundary. However, we apply the idea inductively, again showing that $\frac{\Delta p}{\Delta a} \leq 4$ when adding a square to \mathcal{H} .

To begin, we subdivide the boundary of H_{n+1} into (necessarily finitely many) nonoverlapping line segments of three types.

- P_0 These segments (possibly degenerate) are maximal subsegments of $\partial H_{n+1} \cap \mathcal{H}$. Two may intersect, but only at a vertex of H_{n+1} .
- P_1 A maximal segment $S \subset \partial H_{n+1} \setminus \mathcal{H}$ is a P_1 segment if any line that intersects S and is orthogonal to S also intersects $\mathcal{H} \cap H_{n+1}$.

P_2 These are the remaining maximal segments $S \subset \partial H_{n+1} \setminus \mathcal{H}$ and are characterized by the property that any line that intersects such a segment and is orthogonal to it misses $\mathcal{H} \cap H_{n+1}$.

Note that the mirror image of a P_2 segment is another P_2 segment, while the mirror image of a P_1 segment is always contained in a P_2 segment. The mirror image of a point in a P_0 segment can be either a P_0 point or a P_1 point. See Figure 5.

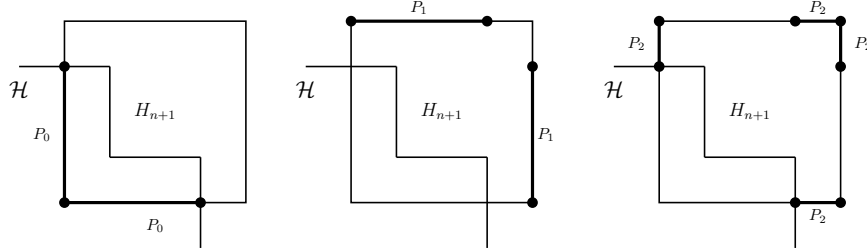


Figure 7: Boundary Segments on H_{n+1}

It is clear that no P_0 segment contributes Δp , but what is also true is that P_1 segments also contribute no net perimeter to Δp .

To see this, suppose S is a P_1 segment and let S^* be the mirror image of S on ∂H_{n+1} . Then $S^* \subset P_0$ and consequently, there is a portion of $\partial \mathcal{H} \cap H_{n+1}$ that covers S in its entirety and this portion of the perimeter of \mathcal{H} is no longer part of the boundary of $\mathcal{H} \cup H_{n+1}$. Thus, although every P_1 segment contributes to the boundary of $\mathcal{H} \cup H_{n+1}$, that addition is balanced by a subtraction from $\partial \mathcal{H}$. That is, P_1 segments create no net perimeter.

Hence, only P_2 segments contribute net perimeter to Δp . Since P_2 segments mirror each other on ∂H_{n+1} , any pair of P_2 segments of individual length b will contribute $2b$ toward Δp . The region between a mirror pair of P_2 segments of length b necessarily misses \mathcal{H} and has area b . However, it is not necessarily true that this strip contributes an area of b to Δa as it is possible to have P_2 segments on both the horizontal and vertical sides of H_{n+1} . In this case, the horizontal and vertical strips intersect. See Figure 5.

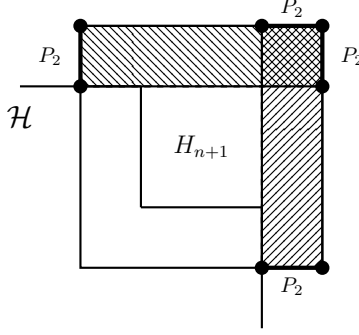


Figure 8: Intersecting Boundary Strips

Suppose there are α mirrored pairs of horizontal P_2 segments that have (individual) lengths h_1, \dots, h_α . Likewise, suppose there are β pairs of vertical P_2 segments that have lengths v_1, \dots, v_β and set

$$h = \sum_{i=1}^{\alpha} h_i \quad \text{and} \quad v = \sum_{i=1}^{\beta} v_i.$$

By inclusion/exclusion, the contribution to Δa of all of the strips is $h + v - hv$, and so

$$\Delta a \geq h + v - hv \geq h + v - \left(\frac{h + v}{2} \right)^2.$$

Since the P_2 segments are the only segments making a net positive contribution to Δp , $\Delta p \leq 2(h + v)$ and hence,

$$\frac{\Delta p}{\Delta a} \leq \frac{2(h + v)}{h + v - hv}. \quad (5)$$

Computing the maximum of (5) it is easy matter to see that $\frac{2(h+v)}{h+v-hv} \leq 4$ when $h, v \in (0, 1]$. In fact, (5) only obtains its maximum value of 4 when $h = v = 1$. This computation then completes the inductive proof and this section. \square

6 The Non-oriented Case

6.1 Need to Think Outside the Box

The method of looking at $\frac{\Delta p}{\Delta a}$ when adding an $n+1$ st square fails for non-oriented squares. Here is a simple counterexample.

Example 3. In the figure below, H intersects H_{n+1} covering all but one small triangle in the bottom left corner of H_{n+1} with base and height b .

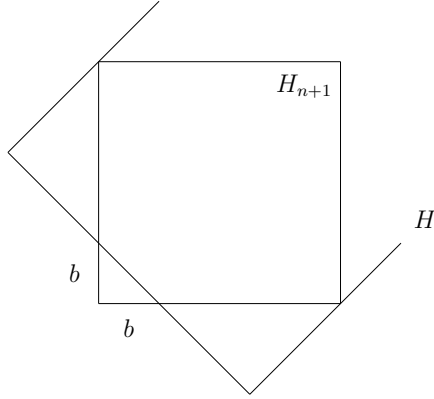


Figure 9: A Counterexample

Then, $\frac{\Delta p}{\Delta a}$ is easily computed.

$$\frac{\Delta p}{\Delta a} = \frac{2b - \sqrt{2}b}{b^2/2} = \frac{4 - 2\sqrt{2}}{b}$$

By selecting b sufficiently small, $\frac{\Delta p}{\Delta a}$ may be made arbitrarily large. Therefore, the method of showing $\frac{\Delta p}{\Delta a} \leq 4$ when adding a square fails for non-oriented squares.

6.2 Being Centered Helps

While this subsection does not directly address our problem, its result will be used in the following section. This result and proof are also presented in [[4]]. However, we are unsure if this paper is the first place this result appears. In any case, our presentation closely follows Z. Gyenes's.

Theorem 4. *The perimeter to area ratio of n unit squares centered on the same location is 4.*

Proof. Let \mathcal{H} be the union of squares where all component squares share the same center point. We partition \mathcal{H} into disjoint triangles by adding line segments between the common center and vertices of the H_i 's as well as the vertices created by the intersection of any two of the H_i 's. The height of any resulting triangle is $\frac{1}{2}$ since the base of any such triangle lies on the boundary of a single H_i . See Figure 10. Summing the areas of all these triangles, we find that $a(\mathcal{H}) = \frac{1}{4}p(\mathcal{H})$ and thus, $\frac{p(\mathcal{H})}{a(\mathcal{H})} = 4$. \square

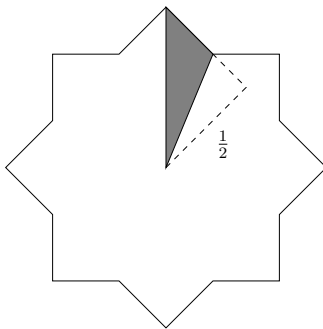


Figure 10: One Partitioning Triangle in a Figure with 2 Squares

7 Counterexample? Pack Your Squares Tightly

In this section, we examine the nature of a potential counterexample to the *PAC* and show that any possible counterexample is constrained in a strong way. We then prove a variant of Theorem 1 where circles rather than squares are stacked.

Because of the finite nature of the problem, if a counterexample exists there will be a counterexample with a minimal number of squares. We focus our attention on this minimal counterexample. Throughout this section, \mathcal{H} will be a finite union of squares such that $\frac{p(\mathcal{H})}{a(\mathcal{H})} > 4$ and for any i ,

$$\frac{p(\mathcal{H} \setminus H_i)}{a(\mathcal{H} \setminus H_i)} \leq 4.$$

We refer to such a counterexample as *optimal*. The following theorem shows that in such an optimal counterexample, any individual component square must share a large portion of its area with the rest of the figure.

Theorem 5. *If $\mathcal{H} = \bigcup_{i=1}^n H_i$ is an optimal counterexample, then for each $i \leq n$, $a(H_i \cap (\mathcal{H} \setminus H_i)) > \frac{\pi}{4}$.*

Proof. Using the argument of the proof of Theorem 1 in Section 4, we assume $\frac{\Delta p}{\Delta a} > 4$ when removing any square, H_i from \mathcal{H} .

We maximize $\frac{\Delta p}{\Delta a}$ for a fixed Δa . To simplify notation, for a fixed i , define

$$\alpha = a(H_i \cap (\mathcal{H} \setminus H_i))$$

In the case of removing the square H_i , $\Delta a = 1 - \alpha$. When we remove the square H_i from \mathcal{H} , a portion of the perimeter of H_i is removed. However, some of this perimeter might have been covered by $\mathcal{H} \setminus H_i$ and some perimeter of $\mathcal{H} \setminus H_i$ that was covered by H_i might be revealed. Between the perimeter of $\mathcal{H} \setminus H_i$ that is revealed and the perimeter of H_i that was covered by \mathcal{H} , there must be at least enough length to enclose the area that is left behind when H_i is removed.

Thus, $\Delta p \leq 4 - x$ where x is the minimal perimeter required to enclose an area of α inside an unit square (if the square boundary is used to enclose the area, it contributes to x).

The isoperimetric inequality states that, in general the minimum perimeter needed to enclose a fixed area in a plane is given by a circle. This fact holds provided a circle of area α can fit inside of the unit square. It is easily shown that a circle of area α can fit inside a unit square so long as $\alpha \leq \frac{\pi}{4}$. Then, assuming $0 \leq \alpha \leq \frac{\pi}{4}$,

$$\frac{\Delta p}{\Delta a} \leq \frac{4 - 2\sqrt{\pi\alpha}}{1 - \alpha}$$

From this it is easy to verify that $\frac{\Delta p}{\Delta a} \leq 4$ when $0 \leq \alpha \leq \frac{\pi}{4}$ and hence, H_i must share more than $\frac{\pi}{4}$ of its area with H , as claimed. \square

One generalization of the *PAC* is to examine the union of finitely copies of different fixed sets. The method just used gives a proof of the *PAC* for circles.

Corollary 6. *The perimeter to area ratio of the union of finitely many circles of radius 1 in the plane does not exceed 2.*

Proof. Applying the exact same argument as in the proof of Theorem 5 shows the result. \square

8 Conclusion

Keleti's *Perimeter to Area Conjecture* is particularly intriguing in a number of ways. First, a full generalization to convex sets is not true so that any appropriate generalized version must involve a parsing of the convex sets in some, presumably geometric way. But what this might be is a mystery. Second is the simplicity of the conjecture itself. The fact that this conjecture is not settled seems to show that we are missing some important geometric fact concerning unions of square regions. Finally, it is interesting that disparate approaches to the problem can provide new clues. For example, the method of inclusion/exclusion yields the result that the "virgin" area of each square of an optimal counterexample can be no more than $1 - \pi/4$.

In a subsequent paper, [6] we study the differentiability properties of both the perimeter and area functions of a union of n unit squares in the plane.

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